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Riemann–Hilbert problems for the Ernst equation and fibre bundles

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Abstract

Riemann–Hilbert techniques are used in the theory of completely integrable differential equations to generate solutions that contain a free function which can be used at least in principle to solve initial or boundary-value problems. The solution of a boundary-value problem is thus reduced to the identification of the jump data of the Riemann–Hilbert problem from the boundary data. But even if this can be achieved, it is very difficult to get explicit solutions since the matrix Riemann–Hilbert problem is equivalent to an integral equation. In the case of the Ernst equation (the stationary axisymmetric Einstein equations in vacuum), it was shown in a previous work that the matrix problem is gauge equivalent to a scalar problem on a Riemann surface. If the jump data of the original problem are rational functions, this surface will be compact which makes it possible to give explicit solutions in terms of hyperelliptic theta functions. In the present work, we discuss Riemann–Hilbert problems on Riemann surfaces in the framework of fibre bundles. This makes it possible to treat the compact and the non-compact case in the same setting and to apply general existence theorems.

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1. Introduction

Riemann–Hilbert techniques provide a powerful tool if one wants to solve initial or boundary-value problems for completely integrable differential equations. They are used to generate solutions with a prescribed singularity structure that contain a free function. In the

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case of a boundary-value problem, this function has to be chosen in a way that the solution takes the prescribed values at the given boundary, and similarly for initial value problems. Whereas this does not pose any problems in principle, there is little hope in practice to get explicit solutions to boundary-value problems in this way. The reason for this is that the free functions (the 'jump data' of the Riemann–Hilbert problem) enter the solutions of the differential equation as part of an integral equation from which the solutions have to be constructed. This implies that one has to solve this integral equation first on the boundary where the boundary data are prescribed in order to determine the jump data. In a second step, one has to solve the integral equation with the then fixed jump data in the region under consideration. In general, both steps cannot be done explicitly.

In the case of the stationary axisymmetric Einstein equations, the situation is however different. The typical problem one has to consider there is the exterior of a relativistic star or a galaxy. Within these models, the matter leads to boundary conditions for the vacuum equations if a solution of the Einstein equations in the matter region is known. Special cases are two-dimensionally extended matter distributions where the field equations reduce to ordinary differential equations, e.g., disks that are discussed in astrophysics as models for certain galaxies. Thus one has to consider boundary-value problems for the vacuum equations in the case of compact matter distributions where the boundary is the surface of the matter, and where the boundary data follow from the metric functions in the matter region. Since the vacuum equations are equivalent to a single complex differential equation, the Ernst equation [1,2], which is completely integrable [3–5], one can use Riemann–Hilbert techniques to solve the resulting boundary-value problems.

In a previous work [6], it was shown that Riemann–Hilbert problems for the Ernst equation with analytic jump functions are gauge equivalent to a scalar problem on a Riemann surface. In the case of rational jump data, this surface is compact which makes it possible to give explicit solutions to the Ernst equation in terms of hyperelliptic theta functions. Thus it is not necessary to consider integral equations in this case. The physical properties of the resulting class of solutions were discussed in [7–9] where it was shown that the solutions can have the expected regularity properties and asymptotic behaviour. In the present article, we discuss the Riemann–Hilbert problems on Riemann surfaces in the framework of fibre bundles which makes it possible to treat the case of compact and non-compact Riemann surfaces within the same setting. Using a theorem of Röhrl [10], we obtain an existence proof for the solutions to the Riemann–Hilbert problems. In the case of non-compact Riemann surfaces, the constructed bundles are trivial due to Grauert's theorem [11].

The paper is organized as follows: in Section 2, we recall that the Ernst equation can be treated as the integrability condition for an overdetermined linear differential system, for which we formulate the matrix Riemann–Hilbert problem. As an example we consider the scalar problem in the complex plane which can be solved with the help of the Plemelj formula [12]. The matrix problem for the Ernst equation is equivalent to an integral equation which cannot be solved explicitly in general. Explicit solutions can in general only be obtained if the jump matrix is diagonal thus implying that the resulting solutions are static. In this particular case the Ernst equation reduces to the axisymmetric Laplace equation. By the help of gauge transformations of the linear system of the Ernst equation, we are able to transform the matrix problem to a scalar one on a Riemann surface in Section 3. We discuss the relation between fibre bundles and Riemann–Hilbert problems on Riemann surfaces. These results are used to prove the existence for the solutions of the original problem. In the case of non-compact Riemann surfaces, Grauert's theorem [11] implies that the constructed bundles are trivial. On compact surfaces, we recover the explicit solutions in terms of theta functions.

2. The Riemann–Hilbert problem for the Ernst equation

It is well known that the vacuum metric in the stationary and axisymmetric case can be written in the Weyl–Lewis–Papapetrou form (see [13])

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}(\mathrm{d}t + a\,\mathrm{d}\phi)^2 + \mathrm{e}^{-2U}(\mathrm{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2) + \rho^2\,\mathrm{d}\phi^2), \tag{2.1}$$

where ρ and ζ are Weyl's canonical coordinates and ∂_t and ∂_ϕ are the two commuting (asymptotically) timelike, respectively, spacelike Killing vectors. In this case, the field equations are equivalent to the Ernst equation for the potential f where $f = \mathrm{e}^{2U} + \mathrm{i}b$, and where the real function b is related to the metric functions via $b_z = -(\mathrm{i}/\rho)\mathrm{e}^{4U}a_z$. Here the complex variable z stands for $z = \rho + \mathrm{i}\zeta$. With these settings, the Ernst equation reads

$$f_{z\bar{z}} + \frac{1}{2(z+\bar{z})}(f_{\bar{z}} + f_z) = \frac{2}{f+\bar{f}} f_z f_{\bar{z}}, \tag{2.2}$$

where a bar denotes complex conjugation in $\bar{\mathbb{C}}$. With a solution of the Ernst equation, the metric function U follows directly from the definition of the Ernst potential whereas a and k can be obtained from f via quadratures.

The importance of the formulation of the field equations in terms of the Ernst potential arises from the fact that the Ernst equation is completely integrable, see [3–5]. This means that the latter can be treated as the integrability condition of an overdetermined linear differential system that contains an additional complex parameter, the so called spectral parameter, that reflects an underlying symmetry of the Ernst equation. We use the linear system for the 2×2-matrix Φ of [5],

$$\Phi_z(K,z,\bar{z}) = \left\{\begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} + \frac{K - \mathrm{i}\bar{z}}{\mu_0}\begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix}\right\}\Phi(K,z,\bar{z}), \tag{2.3}$$

$$\Phi_{\bar{z}}(K,z,\bar{z}) = \left\{\begin{pmatrix} \bar{M} & 0 \\ 0 & \bar{N} \end{pmatrix} + \frac{K + \mathrm{i}z}{\mu_0}\begin{pmatrix} 0 & \bar{M} \\ \bar{N} & 0 \end{pmatrix}\right\}\Phi(K,z,\bar{z}). \tag{2.4}$$

Here the spectral parameter K resides on a family of Riemann surfaces $\mathcal{L}(z,\bar{z})$ parametrized by the physical coordinates z and $\bar{z}$ and given by $\mu_0^2(K) = (K - \mathrm{i}\bar{z})(K + \mathrm{i}z)$. A point on $\mathcal{L}$ is denoted by $P = (K, \mu_0(K))$ with $K \in \bar{\mathbb{C}}$. The functions M and N depend only on z and $\bar{z}$ but not on K, and have the form

$$M = \frac{f_z}{f+\bar{f}}, \quad N = \frac{\bar{f}_z}{f+\bar{f}}, \tag{2.5}$$

where f is again the Ernst potential.

To construct solutions to the Ernst equation with the help of the above linear system, one investigates the singularity structure of the matrices $\Phi_z\Phi^{-1}$ and $\Phi_{\bar{z}}\Phi^{-1}$ with respect to the spectral parameter and infers a set of conditions for the matrix Φ that satisfies the linear system (2.3) and (2.4). This is done (see e.g. [14]) in:

Theorem 2.1. *Let Φ be subject to the following conditions:*

I. *$\Phi(P)$ is holomorphic and invertible at the branch points $P_0 = -\mathrm{i}z$ and $\bar{P}_0$ such that the logarithmic derivative $\Phi_z\Phi^{-1}$ diverges as $(K+\mathrm{i}z)^{-1/2}$ at P_0 and $\Phi_{\bar{z}}\Phi^{-1}$ as $(K-\mathrm{i}\bar{z})^{1/2}$ at $\bar{P}_0$.*

II. *All singularities of Φ on $\mathcal{L}$ (poles, essential singularities, zeros of the determinant of Φ, branch cuts and branch points) are regular which means that the logarithmic derivatives $\Phi_z\Phi^{-1}$ and $\Phi_{\bar{z}}\Phi^{-1}$ are holomorphic there.*

III. *Φ is subject to the reduction condition*

$$\Phi(P^\sigma) = \sigma_3\Phi(P)\sigma_1, \tag{2.6}$$

where σ is the involution on $\mathcal{L}$ that interchanges the sheets, and σ_1 and σ_3 are Pauli matrices.

IV. *The normalization and reality condition*

$$\Phi(P=\infty^+) = \begin{pmatrix} \bar{f} & 1 \\ f & -1 \end{pmatrix} \tag{2.7}$$

is fulfilled.

Then the function f in (2.7) is a solution to the Ernst equation.

This theorem has the following:

Corollary 2.2. *Let $\Phi(P)$ be a matrix subject to the conditions of Theorem 2.1 and $C(K)$ be a 2×2-matrix that only depends on $K \in \bar{\mathbb{C}}$ with the properties (the α_i are scalar functions)*

$$\begin{aligned} &C(K) = \alpha_1(K)\hat{1} + \alpha_2(K)\sigma_1, \\ &\alpha_1(\infty) = 1, \quad \alpha_2(\infty) = 0. \end{aligned} \tag{2.8}$$

Then the matrix $\Phi'(P) = \Phi(P)C(K)$ also satisfies the conditions of Theorem 2.1 and $\Phi'(\infty^+) = \Phi(\infty^+)$.

In other words, matrices Φ which are related through the multiplication from the right by a matrix C of the above form lead to the same Ernst potential though their singularity structure may be vastly different (the functions α_i need not be holomorphic). Therefore this multiplication is called a gauge transformation.

Theorem 2.1 can be used to construct solutions to the Ernst equation by determining the structure and the singularities of Φ in accordance with conditions I–IV. In the present paper, we will concentrate on the Riemann–Hilbert problem for the Ernst equation which can be formulated in the following form: Let Γ be a set of (orientable piecewise smooth) contours $\Gamma_k \subset \mathcal{L}$ $(k = 1,\dots,l)$ such that with $P \in \Gamma$ also $\bar{P} \in \Gamma$ and $P^\sigma \in \Gamma$. Let

$\mathcal{G}_k(P)$ be matrices on Γ_k with analytic components and non-vanishing determinant subject to the reality condition $\mathcal{G}_{ii}(\bar{P}) = \bar{\mathcal{G}}_{ii}(P)$ for the diagonal elements, and $\mathcal{G}_{ij}(\bar{P}) = -\bar{\mathcal{G}}_{ij}(P)$ for the offdiagonal elements. We define $\gamma(t, \Gamma_j) = 1$ if $t \in \Gamma_j$ and 0 otherwise and $\mathcal{G} = \sum_{k=1}^{l} \gamma(t, \Gamma_k)\mathcal{G}_k$. Let $\mathcal{G}(P^\sigma) = \sigma_1 \mathcal{G}(P)\sigma_1$. Both Γ and $\mathcal{G}$ have to be independent of $z, \bar{z}$. The matrix Φ has to be everywhere regular except at the contour Γ where the boundary values on both sides of the contours (denoted by $\Phi_\pm$) are related via

$$\Phi_-(P) = \Phi_+(P)\mathcal{G}_i(P)|_{P \in \Gamma_i}\,. \tag{2.9}$$

It may be easily checked that a matrix Φ constructed in this way satisfies the conditions of Theorem 2.1. Furthermore, it can be seen from the theorem that the only possible singularities of the Ernst potential can occur where the conditions are not satisfied, i.e. where Φ cannot be normalized or where P_0 coincides with one of the singularities of Φ, in our case the contour Γ. The latter makes the Riemann–Hilbert problem very useful if one wants to solve boundary-value problems for the Ernst equation: choose the contour Γ in a way that $P_0 \in \Gamma$ just corresponds to the contour in the meridian $(z, \bar{z})$-plane where the boundary values are prescribed. The Ernst potential will in general not be continuous at this contour, but its boundary values will be bounded. Notice however that the Ernst potential will not always be singular if P_0 coincides with a singularity of Φ since the latter may be e.g. a pure gauge. Theorem 2.1 merely ensures that the solution will be regular at all other points.

To solve Riemann–Hilbert problems, one basically uses the same methods as in the simplest case, the problem in the complex plane for a scalar function ψ, see e.g. [15]. If Γ_K is a simply connected closed smooth contour and G a non-zero Hölder continuous function on Γ_K in $\mathbb{C}$, the function ψ that is holomorphic except at the contour Γ_K, where

$$\psi_- = \psi_+ G \tag{2.10}$$

is obviously given by the Cauchy integral,

$$\psi(K) = F(K) \exp\left(\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_K} \frac{\ln G \,\mathrm{d}X}{X - K} \right), \tag{2.11}$$

where $F(K)$ is an arbitrary holomorphic function, and where the principal value of the logarithm has to be taken. The well-known analytic properties of the Cauchy integral ensure that condition (2.10) is satisfied. Formula (2.11) shows that the solution to a Riemann–Hilbert problem of the above form is only determined up to a holomorphic function. Since F is holomorphic, the solution will be uniquely determined (due to Liouville's theorem) by a normalization condition $\psi(\infty) = \psi_0$ for $\infty \notin \Gamma_K$. Uniqueness is lost if one allows for additional poles since F in (2.11) has then to be replaced by a meromorphic function. We note that the above conditions on the contour may be relaxed: Γ may consist of a set of piecewise smooth orientable contours which are not closed. The Plemelj formula, see e.g. [12], assures that formula (2.11) still gives the solution to (2.10). A normalization condition will however only establish uniqueness of the solution if $G = 1$ at the endpoints of Γ, i.e. if the index of the problem (2.10) vanishes.

Riemann–Hilbert problems on the sphere $\mathcal{L}$ which occur in the case of the Ernst equation can be treated in much the same way as the problems in the complex plane. The basic building block for the solutions is the differential of the third kind $\mathrm{d}\omega_{K^+K^-}(X)$ that corresponds to the differential $\mathrm{d}X/(X-K)$ in the complex plane, i.e. a differential that can be locally written as $F(X,K)\,\mathrm{d}X$, where $F(X,K)$ is holomorphic except for $X=K^\pm$ and where the residues are ± 1 respectively. If we make the ansatz

$$\Phi - \Phi_0 = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \chi(X)\,\mathrm{d}\omega_{K^+K^-}(X), \tag{2.12}$$

where the 2×2-matrix χ is given by $\chi = \sum_{k=1}^{l} \gamma(t,\Gamma_k)\chi_k$ and where Φ_0 is holomorphic, we get for (2.9) at the contour Γ with the Plemelj formula $\Phi^\pm = \pm\frac{1}{2}\chi + \Phi_0 + (1/2\pi\mathrm{i})\int_\Gamma \chi(X)\,\mathrm{d}\omega_{K^+K^-}(X)$. Thus the Riemann–Hilbert problem (2.9) is equivalent to the integral equations (at Γ)

$$\frac{1}{2}\chi(\mathcal{G}+1) + \frac{1}{2\pi\mathrm{i}} \int_\Gamma \chi(X)\,\mathrm{d}\omega_{K^+K^-}(X)(\mathcal{G}-1) = 0. \tag{2.13}$$

For simplicity, we will only consider the case where the projection of the contour Γ into the complex plane has a simply connected component Γ_K. In [6] we have shown that the general problem (2.9) is gauge equivalent to a problem with

$$\mathcal{G}_1 = \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix}$$

on Γ_1 which is the contour in the + sheet. This gauge transformation does not change the singularity structure of Φ (i.e. Φ will only be singular at Γ). The reduction condition III of Theorem 2.1 implies that

$$\mathcal{G}_2 = \begin{pmatrix} 1 & \beta \\ 0 & \alpha \end{pmatrix}$$

on the contour Γ_2 in the −-sheet. Because of the reality conditions for $\mathcal{G}$, this implies that solutions to the Ernst equation that follow from (2.9) contain two real-valued functions which correspond to α and β. The reduction and reality properties of Φ (see Theorem 2.1) make it possible to consider only one component of the matrix, e.g. Φ_{12} from which the Ernst potential follows as $f = \Phi_{12}(\infty^+)$. With $\Phi_{12} = \psi_0 + (1/4\pi\mathrm{i})\int_{\Gamma_1}\{(\mu_0(K)+\mu_0(X))/(2\mu_0(X)(X-K))\}Z(X,z,\bar{z})\,\mathrm{d}X$ we obtain the Ernst potential for given α and β where Z is the solution of the integral equation

$$\begin{aligned} -\frac{\alpha+1}{2}Z = {}& \frac{\alpha-1}{4\pi\mathrm{i}} \int_{\Gamma_1} \frac{\mu_0(K)+\mu_0(X)}{\mu_0(X)(X-K)} Z(X,z,\bar{z})\,\mathrm{d}X \\ & - \frac{\beta}{4\pi\mathrm{i}} \int_{\Gamma_1} \frac{\mu_0(X)-\mu_0(K)}{\mu_0(X)(X-K)} Z(X,z,\bar{z})\,\mathrm{d}X, \end{aligned} \tag{2.14}$$

and where ψ_0 follows from the normalization condition $\Phi_{12}(\infty^-) = 1$.

Explicit solutions can in general only be obtained for diagonal $\mathcal{G}$, i.e. for $\beta = 0$. In this case we get with the above formulas $f = \bar{f} = \mathrm{e}^{2U}$ with

$$U = -\frac{1}{4\pi \mathrm{i}} \int_{\Gamma_1} \frac{\ln \alpha}{\sqrt{(K-\zeta)^2+\rho^2}}\, \mathrm{d}K. \tag{2.15}$$

Thus all solutions are real in this case which implies that they belong to the static Weyl class. Since the Ernst equation reduces to the axisymmetric Laplace equation for U if f is real, the function U in (2.15) solves the Laplace equation. In fact one can show that the contour integral there is equivalent to the Poisson integral with a distributional density. It can also be directly seen from expression (2.15) that the dependence on the physical coordinates ρ and ζ enters through the branch points of the family of surfaces $\mathcal{L}$.

3. Riemann–Hilbert problems on Riemann surfaces and vector bundles

In Section 2 we recalled that matrix Riemann–Hilbert problems cannot be solved in general. Only for particular cases one can find an explicit form of the solution. In [6] we have shown that in the context of the Ernst equation it is, however, possible to go one step further if one drops the condition that the gauge transformed matrix Φ' has the same singularity structure as the original matrix Φ in (2.9). It was furthermore shown there that the Riemann–Hilbert problem (2.9) is gauge equivalent to a problem with diagonal matrix $\mathcal{G}' = \mathrm{diag}(G, 1)$ on a two-sheeted covering $\hat{\mathcal{L}}$ of $\mathcal{L}$, given by an equation of the form

$$\hat{\mu}^2(K) = F(K)\, H(K), \tag{3.1}$$

where $F(K)$ and $H(K)$ are holomorphic functions. They follow from the jump matrix $\mathcal{G}$ via

$$\frac{F(K)}{H(K)} = \frac{(\mathcal{G}_{11}-\mathcal{G}_{12}+\mathcal{G}_{21}-\mathcal{G}_{22})(\mathcal{G}_{11}-\mathcal{G}_{12}-\mathcal{G}_{21}+\mathcal{G}_{22})}{(\mathcal{G}_{11}+\mathcal{G}_{12}-\mathcal{G}_{21}-\mathcal{G}_{22})(\mathcal{G}_{11}+\mathcal{G}_{12}+\mathcal{G}_{21}+\mathcal{G}_{22})}, \tag{3.2}$$

whereas the analytic jump function G can be expressed via the components of the original jump matrix $\mathcal{G}$ by

$$\frac{G+1}{G-1} = \sqrt{\frac{(\mathcal{G}_{11}-\mathcal{G}_{12}-\mathcal{G}_{21}+\mathcal{G}_{22})(\mathcal{G}_{11}+\mathcal{G}_{12}+\mathcal{G}_{21}+\mathcal{G}_{22})}{(\mathcal{G}_{11}-\mathcal{G}_{12}+\mathcal{G}_{21}-\mathcal{G}_{22})(\mathcal{G}_{11}+\mathcal{G}_{12}-\mathcal{G}_{21}-\mathcal{G}_{22})}}. \tag{3.3}$$

By definition a Riemann surface is given by an equation of the form $f(K, \mu) = 0$, where $f(K, \mu)$ is an analytic function of K and μ. If $f(K, \mu)$ is a polynomial in both variables one speaks of a compact Riemann surface. Therefore, for our purposes it is sufficient to solve a scalar Riemann–Hilbert problem on a four-sheeted Riemann surface which is, dependent on the initial variables $\mathcal{G}_{ij}$, either compact or non-compact (if the components of $\mathcal{G}$ are rational functions, the surface will be compact).

In the mentioned paper we restricted ourselves to the case of $\hat{\mathcal{L}}$ being compact with genus g, where it is possible to give explicit solutions to the Riemann–Hilbert problem in terms of theta functions. Here we give a characterization of solutions to Riemann–Hilbert problems in terms of fibre bundle theory, which allows for a treatment of both the compact and non-compact case.

It is a well known fact, see [16], that there is a relation between Riemann–Hilbert problems on Riemann surfaces and holomorphic vector bundles over them. To make the paper self-consistent, we begin with a brief introduction into fibre bundle theory, see [17,18].

Let us recall what a *differentiable fibre bundle* is.

Definition 3.1. A *differentiable fibre bundle* (E, M, F, G, π) is a 5-tuple consisting of

I. A differentiable manifold E – the so called *total space*.
II. A differentiable manifold M – the so called *base space*.
III. A differentiable manifold F – the *fibre* or *typical fibre*.
IV. A surjective map $\pi : E \to M$, called *projection*. The inverse image $\pi^{-1}(p) \equiv F_p \simeq F$ of $p \in M$ is called the *fibre* at p.
V. A Lie group G – the *structure group* – acting on F on the left.
VI. An open covering $\{U_i\}$ of M together with diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$, such that $\pi \circ \phi_i(p, f) = p$ $(p \in U_i, f \in F)$. We call ϕ_i the *local trivialisation* since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$.
VII. If we set $\phi_{i,p}(f) \doteq \phi_i(p, f)$ then $\phi_{i,p} : F \to F_p$ is a diffeomorphism. If $U_i \cap U_j \neq \emptyset$ we require that $t_{ij}(p) \doteq \phi_{i,p}^{-1}\phi_{j,p} : F \to F$ be an element of the structure group G. Then ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ as

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f). \tag{3.4}$$

We call the $\{t_{ij}\}$ the *transition functions*.

Let us say a few words about this definition. If we take a chart U_i of the base space M then $\pi^{-1}(U_i)$ is diffeomorphic to $U_i \times F$ with diffeomorphism $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$. If the intersection $U_i \cap U_j \neq \emptyset$, there are two maps ϕ_i and ϕ_j on this intersection. Let $u \in E$ such that $\pi(u) = p \in U_i \cap U_j$. We then have

$$\phi_i^{-1}(u) = (p, f_i), \qquad \phi_j^{-1}(u) = (p, f_j).$$

There is a map $t_{ij} : U_i \cap U_j \to G$ which relates f_i and f_j: $f_i = t_{ij}(p)f_j$.

The functions $t_{ij}(p)$ cannot be chosen arbitrarily in order to be the transition functions of a fibre bundle. They have to satisfy some consistency conditions:

$$\begin{aligned} t_{ii}(p) &= \text{ identity map} \quad (p \in U_i) \\ t_{ij}(p) &= t_{ji}(p)^{-1} \quad (p \in U_i \cap U_j) \\ t_{ij}(p)t_{jk}(p) &= t_{ik}(p) \quad (p \in U_i \cap U_j \cap U_k). \end{aligned} \tag{3.5}$$

A fibre bundle is called *trivial*, if all the transition functions can be taken to be identity maps. A trivial bundle is of the form $E = M \times F$.

The transition functions are so important because they contain all the information needed to construct a fibre bundle. Let us now show how starting from a 5-tuple $(M, \{U_i\}, \{t_{ij}(p)\},$

F, G) a fibre bundle over M with typical fibre F can be constructed. Finding the bundle means finding unique E, π and ϕ_i from the above data. We define

$$X \equiv \bigcup_i (U_i \times F), \tag{3.6}$$

and introduce on X an equivalence relation $\sim$ as follows. We say that $(p, f) \in U_i \times F$ and $(q, f') \in U_j \times F$ are equivalent, $(p, f) \sim (q, f')$, if and only if $p = q$ and $f' = t_{ij}(p) f$. The total space E of the fibre bundle is then defined by

$$E = X / \sim . \tag{3.7}$$

We denote an element of E by the equivalence class $[(p, f)]$. The projection $\pi : E \to M$ is given by

$$\pi : [(p, f)] \to p, \tag{3.8}$$

and the local trivialisation $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ is given by

$$\phi_i : (p, f) \to [(p, f)], \tag{3.9}$$

with $p \in U_i$ and $f \in F$. The above data reconstruct the bundle E uniquely.

Let us now make the relation between Riemann–Hilbert problems on Riemann surfaces and vector bundles on them more explicit.

3.1. The compact case

First we will consider the case that the Riemann surface $\hat{\mathcal{L}}$ obtained as the double covering via the procedure described in [6] of the Riemann surface $\mathcal{L}$ is compact, i.e. we consider the Riemann–Hilbert problem

$$\phi_-(P) = \phi_+(P) G(P) \tag{3.10}$$

on a compact Riemann surface. Due to the fact that on such surfaces theta functions are the basic building blocks for the construction of meromorphic functions, see [19], we may express the solution to (3.10) in terms of these functions. But, in order to make contact with the case of $\hat{\mathcal{L}}$ being non-compact, we describe here how a solution to (3.10) is connected with some line bundle over $\hat{\mathcal{L}}$. To this end we want to make use of the above result that a line bundle over a manifold $\hat{\mathcal{L}}$ is completely determined by a triple $(\hat{\mathcal{L}}, \{U_i\}, \{t_{ij}\})$, because (in this case) we have $F = \mathbb{C}$, $G = \mathbb{C}^*$. Let $\{U_i\}$ be a covering of $\hat{\mathcal{L}}$ and $\{\phi_i(P)\}$ solutions to (3.10) in the domains U_i, different from zero. In the domains U_i the original scalar Riemann–Hilbert problem is reduced to a problem on the complex plane $\mathbb{C}$, which can be solved (see the discussion in Section 2).

In other words, the functions $\phi_i(P)$ are non-vanishing on U_i and fulfil (3.10) on the intersection $\Gamma \cap U_i$. If $\Gamma \cap U_i = \emptyset$ then $\phi_i(P)$ is a holomorphic function in U_i. Let us now define some functions $t_{ij}^{\pm}(P)$ in $U_i \cap U_j$ by

$$t_{ij}^{\pm}(P) = \frac{\phi_{j\pm}(P)}{\phi_{i\pm}(P)}, \tag{3.11}$$

for $P \in U_i \cap U_j$. We have

$$t_{ij}^{-}(P) = \frac{\phi_{j-}(P)}{\phi_{i-}(P)} = \frac{\phi_{j+}(P)G(P)}{\phi_{i+}(P)G(P)} = \frac{\phi_{j+}(P)}{\phi_{i+}(P)} = t_{ij}^{+}(P), \tag{3.12}$$

i.e. the functions $t_{ij}(P) \doteq t_{ij}^{+}(P) = t_{ij}^{-}(P)$ do not jump at the contour Γ. It follows immediately from this definition that the t_{ij} fulfil the consistency conditions for transition functions (3.5). Because the $\phi_i(P)$ are non-vanishing the functions $t_{ij}(P)$ take values in $\mathbb{C}^*$ (the complex numbers different from zero). Therefore, we may define a complex bundle with structure group $\mathbb{C}^*$ and standard fibre $\mathbb{C}$, i.e. a line bundle, B_G, over the compact Riemann surface $\hat{\mathcal{L}}$ by the 5-tupel $(\hat{\mathcal{L}}, \{U_i\}, \{t_{ij}(P)\}, \mathbb{C}, \mathbb{C}^*)$. In other words: for $\hat{\mathcal{L}}$ being compact we may associate to a Riemann–Hilbert problem (3.10) a line bundle over this surface.

3.2. The non-compact case

Let us now turn to the case $\hat{\mathcal{L}}$ being non-compact. Contrary to the compact case we do not have the calculus of theta functions associated to a Riemann surface at our disposal in order to construct meromorphic functions. Nevertheless, we may perform a similar construction as in the compact case and construct a vector bundle for a Riemann–Hilbert problem given on this surface. The remarkable point is now that this vector bundle is, due to a theorem by Grauert [11], a trivial one.

To be more precise, let $\hat{\mathcal{L}}$ be equipped with a covering $\mathcal{N} = \{U_i, i \in I\}$, where I denotes some set of indices. Let us suppose that there exists a number N, the covering constant, such that any point $P \in \hat{\mathcal{L}}$ belongs to no more than N domains of the covering, see [16]. We assume that the contour Γ is compact and closed, dividing $\hat{\mathcal{L}}$ into, in general, non-compact domains. To simplify the discussion we are looking for solutions $\phi(P)$ with finite Dirichlet integral, i.e.

$$\int\limits_{\hat{\mathcal{L}}\setminus\Delta(\Gamma)} \overline{\mathrm{d}\phi} \wedge \mathrm{d}\phi < \infty, \tag{3.13}$$

where $\Delta(\Gamma)$ denotes some neighbourhood of Γ. Let $\phi_i(P)$ be, as above, a non-vanishing function on U_i and solving there the Riemann–Hilbert problem (3.10) on the intersection $\Gamma \cap U_i$. If $\Gamma \cap U_i = \emptyset$ then $\phi_i(P)$ is a holomorphic function in U_i. Similarly to the compact case we now define functions $t_{ij}^{\pm}(P)$ in $U_i \cap U_j$ by

$$t_{ij}^{\pm}(P) = \frac{\phi_{j\pm}(P)}{\phi_{i\pm}(P)}. \tag{3.14}$$

Again, we have

$$t_{ij}^{-}(P) = \frac{\phi_{j-}(P)}{\phi_{i-}(P)} = \frac{\phi_{j+}(P)G(P)}{\phi_{i+}(P)G(P)} = \frac{\phi_{j+}(P)}{\phi_{i+}(P)} = t_{ij}^{+}(P), \tag{3.15}$$

for $P \in U_i \cap U_j$, i.e. the functions $t_{ij}(P) \doteq t_{ij}^{+}(P) = t_{ij}^{-}(P)$ do not jump at Γ, analogously to the compact case. These functions also obey the consistency conditions (3.5) and, therefore, we may associate to the Riemann–Hilbert problem (3.10) a vector bundle in the same manner as in the compact case. But, whereas in the compact case one does not know in advance what the global structure of the bundle space B_G looks like, we have in the present case the following:

Theorem 3.2 (Grauert). *Any complex line bundle over a non-compact Riemann surface is trivial.*

Proof. The proof can be found in [11]. □

From this theorem it follows that for non-compact $\hat{\mathcal{L}}$ the line bundle B_G associated the Riemann–Hilbert problem (3.10) has the form

$$B_G \simeq \hat{\mathcal{L}} \times \mathbb{C}. \tag{3.16}$$

To conclude, we have shown that for $\hat{\mathcal{L}}$ being non-compact (where one does not have an explicit solution of the Riemann–Hilbert problem in terms of theta functions) there is a simple geometric characterization of it in terms of fibre bundles over $\hat{\mathcal{L}}$. Due to its local properties the bundle approach allows to reduce the scalar Riemann–Hilbert problems on $\hat{\mathcal{L}}$ to problems on the complex plane $\mathbb{C}$ which can be explicitly solved. For non-compact Riemann surfaces $\hat{\mathcal{L}}$ the resulting total space is given as a direct product whereas in the compact case the total space is, in general, twisted in a non-trivial manner. On the other hand, in this last case, an explicit formulation of the solution in terms of theta functions is possible.

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